Physics 17 Class Notes
Commutation Relations and the Existence of Simultaneous Eigenfunctions

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This writeup illustrates if a wavefunction exists that is simultaneously an eigenstate of two operators $\hat{A}$ and $\hat{B}$, those operators commute,

$$\begin{aligned}\left[\hat{A},\hat{B}\right] &\equiv \hat{A}\hat{B}-\hat{B}\hat{A}\\ &= 0.\end{aligned}$$

We will proceed by considering a wavefunction ψ (which is a function of one or more variables, such as $\mathbf{r}$ and t that I will not bother to write out explicitly here) that is an eigenfunction of operator $\hat{A}$ with eigenvalue a. Mathematically, this means that

$$\hat{A}\psi = a\psi.$$

Now we can consider that the commutator between $\hat{A}$ and $\hat{B}$ is given by

$$\left[\hat{A},\hat{B}\right] = \eta$$

for some (possibly zero) η that is yet to be determined. We can compute

$$\begin{aligned}\left[\hat{A},\hat{B}\right]\psi &= \left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)\psi\\ &= \hat{A}\hat{B}\psi-\hat{B}\hat{A}\psi\\ &= \left(\hat{A}\hat{B}-\hat{B}a\right)\psi\\ &= \left(\hat{A}-a\right)\hat{B}\,\psi, \qquad (1)\end{aligned}$$

which shows that

$$\begin{aligned}\left[\hat{A},\hat{B}\right]\psi &= \left(\hat{A}-a\right)\hat{B}\,\psi\\ \eta\,\psi &= \left(\hat{A}-a\right)\hat{B}\,\psi.\end{aligned}$$

Now let's assume that ψ is also (*i.e.* simultaneously) an eigenfunction of $\hat{B}$ with some eigenvalue b:

$$\hat{B}\psi = B\psi.$$

We can now further simplify from Eq. (1),

$$\begin{aligned}
\left[\hat{A},\hat{B}\right]\psi &= \left(\hat{A}-a\right)\hat{B}\,\psi \\
&= \left(\hat{A}-a\right)b\,\psi \\
&= b\hat{A}\,\psi - ab\,\psi \\
&= (ba-ab)\,\psi \\
&= 0\,\psi.
\end{aligned}$$

So since we've found that

$$\left[\hat{A},\hat{B}\right]\psi = \eta\,\psi$$

and

$$\left[\hat{A},\hat{B}\right]\psi = 0\,\psi,$$

we conclude that $[\hat{A},\hat{B}] = \eta = 0$. If ψ is a simultaneous eigenfunction of $\hat{A}$ and $\hat{B}$, $[\hat{A},\hat{B}] = 0$.