

The first few spherical harmonics

WCC

The spherical harmonics, $Y_{L,M_L}(\boldsymbol{\omega})$, where $\boldsymbol{\omega} = (\theta, \phi)$, up to $L = 4$:

$$\begin{aligned}
Y_{0,0}(\boldsymbol{\omega}) &= \frac{1}{2\sqrt{\pi}} \\
Y_{1,\pm 1}(\boldsymbol{\omega}) &= \mp \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin(\theta) e^{\pm i\phi} \\
Y_{1,0}(\boldsymbol{\omega}) &= \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos(\theta) \\
Y_{2,\pm 2}(\boldsymbol{\omega}) &= \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2(\theta) e^{\pm i2\phi} \\
Y_{2,\pm 1}(\boldsymbol{\omega}) &= \mp \frac{1}{2} \sqrt{\frac{15}{2\pi}} \cos(\theta) \sin(\theta) e^{\pm i\phi} \\
Y_{2,0}(\boldsymbol{\omega}) &= \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2(\theta) - 1) \\
Y_{3,\pm 3}(\boldsymbol{\omega}) &= \mp \frac{1}{8} \sqrt{\frac{35}{\pi}} \sin^3(\theta) e^{\pm i3\phi} \\
Y_{3,\pm 2}(\boldsymbol{\omega}) &= \frac{1}{4} \sqrt{\frac{105}{2\pi}} \cos(\theta) \sin^2(\theta) e^{\pm i2\phi} \\
Y_{3,\pm 1}(\boldsymbol{\omega}) &= \mp \frac{1}{8} \sqrt{\frac{21}{\pi}} (5 \cos^2(\theta) - 1) \sin(\theta) e^{\pm i\phi} \\
Y_{3,0}(\boldsymbol{\omega}) &= \frac{1}{4} \sqrt{\frac{7}{\pi}} (5 \cos^2(\theta) - 3) \cos(\theta) \\
Y_{4,\pm 4}(\boldsymbol{\omega}) &= \frac{3}{16} \sqrt{\frac{35}{2\pi}} \sin^4(\theta) e^{\pm i4\phi} \\
Y_{4,\pm 3}(\boldsymbol{\omega}) &= \mp \frac{3}{8} \sqrt{\frac{35}{\pi}} \sin^3(\theta) \cos(\theta) e^{\pm i3\phi} \\
Y_{4,\pm 2}(\boldsymbol{\omega}) &= \frac{3}{8} \sqrt{\frac{5}{2\pi}} \sin^2(\theta) (7 \cos^2(\theta) - 1) e^{\pm i2\phi} \\
Y_{4,\pm 1}(\boldsymbol{\omega}) &= \mp \frac{3}{8} \sqrt{\frac{5}{\pi}} \sin(\theta) (7 \cos^3(\theta) - 3 \cos(\theta)) e^{\pm i\phi} \\
Y_{4,0}(\boldsymbol{\omega}) &= \frac{3}{16\sqrt{\pi}} (35 \cos^4(\theta) - 30 \cos^2(\theta) + 3).
\end{aligned}$$

The general form is

$$Y_{LM}(\boldsymbol{\omega}) = (-1)^M \sqrt{\frac{2L+1}{4\pi} \frac{(L-M)!}{(L+M)!}} P_L^M [\cos(\theta)] e^{iM\phi}$$

where $P_b^a[x]$ is an associated Legendre polynomial in x .

Real-valued angular functions:

$$u_{LM}(\hat{\mathbf{r}}) \equiv \frac{1}{2} (Y_{LM}(\boldsymbol{\omega}) + Y_{LM}^*(\boldsymbol{\omega}))$$

$$v_{LM}(\hat{\mathbf{r}}) \equiv -\frac{i}{2} (Y_{LM}(\boldsymbol{\omega}) - Y_{LM}^*(\boldsymbol{\omega})).$$

$$u_{0,0}(\hat{\mathbf{r}}) = \frac{1}{2\sqrt{\pi}}$$

$$-\sqrt{2} u_{1,1}(\hat{\mathbf{r}}) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{x}{r}$$

$$-\sqrt{2} v_{1,1}(\hat{\mathbf{r}}) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{y}{r}$$

$$u_{1,0}(\hat{\mathbf{r}}) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{z}{r}$$

$$\sqrt{2} u_{2,2}(\hat{\mathbf{r}}) = \frac{1}{4} \sqrt{\frac{15}{\pi}} (x^2 - y^2) \frac{1}{r^2}$$

$$\sqrt{2} v_{2,2}(\hat{\mathbf{r}}) = \frac{1}{2} \sqrt{\frac{15}{\pi}} \frac{xy}{r^2}$$

$$-\sqrt{2} u_{2,1}(\hat{\mathbf{r}}) = \frac{1}{2} \sqrt{\frac{15}{\pi}} \frac{zx}{r^2}$$

$$-\sqrt{2} v_{2,1}(\hat{\mathbf{r}}) = \frac{1}{2} \sqrt{\frac{15}{\pi}} \frac{zy}{r^2}$$

$$u_{2,0}(\hat{\mathbf{r}}) = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3z^2 - r^2) \frac{1}{r^2}$$