

Harmonic Oscillator Cheat Sheet

$$\begin{aligned}
\hat{x} &= x_0(a^\dagger + a), & x_0 \equiv \sqrt{\frac{\hbar}{2m\omega}} &= \frac{\hbar}{2p_0} &= \frac{p_0}{m\omega} \\
\hat{p} &= ip_0(a^\dagger - a), & p_0 \equiv \sqrt{\frac{\hbar m\omega}{2}} &= \frac{\hbar}{2x_0} &= m\omega x_0 \\
H_0 &= \frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger) = \hbar\omega(a^\dagger a + \frac{1}{2}), & \hat{n} &= a^\dagger a & \\
[a, a^\dagger] &= 1, & aa^\dagger &= 1 + a^\dagger a & \\
[a^\dagger a, a] &= -a, & [a^\dagger a, a^\dagger] &= a^\dagger & \\
[a^\dagger a, a^\dagger + a] &= a^\dagger - a, & [H_0, \hat{x}] &= -i\frac{\hbar}{m}\hat{p} & \\
[a^\dagger a, a^\dagger - a] &= a^\dagger + a, & [H_0, \hat{p}] &= i\hbar m\omega^2 \hat{x} & \\
[a^\dagger + a, a^\dagger - a] &= 2, & [\hat{x}, \hat{p}] &= i\hbar & \\
a|n\rangle &= \sqrt{n}|n-1\rangle, & a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle & \\
a(a^\dagger a)^N &= (1 + a^\dagger a)^N a & & &
\end{aligned}$$

Typically, when we transform into the interaction picture, we can use the Baker-Hausdorff lemma,

$$e^{i\lambda\hat{B}}\hat{A}e^{-i\lambda\hat{B}} = \hat{A} + i\lambda[\hat{B}, \hat{A}] + \frac{(i\lambda)^2}{2!}[\hat{B}, [\hat{B}, \hat{A}]] + \frac{(i\lambda)^3}{3!}[\hat{B}, [\hat{B}, [\hat{B}, \hat{A}]]] + \dots$$

where only the first two terms matter when \hat{B} commutes with the commutator of \hat{A} and \hat{B} . However, this last condition is not true for $\hat{B} \propto \hat{n}$ and $\hat{A} \propto a$. The harmonic oscillator operators in the interaction picture with respect to $H_0 = \omega(a^\dagger a + \frac{1}{2})$ take a little more work, but are given by

$$e^{iH_0 t/\hbar} a e^{-iH_0 t/\hbar} = a e^{-i\omega t} \quad \text{and likewise} \quad e^{iH_0 t/\hbar} a^\dagger e^{-iH_0 t/\hbar} = a^\dagger e^{i\omega t}$$

Coherent states and the displacement operator:

$$\begin{aligned}
a|\alpha\rangle &= \alpha|\alpha\rangle & |\alpha\rangle &\equiv e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
\int d^2\alpha |\alpha\rangle\langle\alpha| &= \pi & \langle\beta|\alpha\rangle &= e^{-\frac{1}{2}(|\beta|^2 + |\alpha|^2 - 2\beta^*\alpha)} \\
\mathcal{D}(\ell, \varphi) &= e^{-i(\ell\hat{p} - \varphi\hat{x})/\hbar} & \alpha(\ell, \varphi) &\equiv \frac{\ell}{2x_0} + i\frac{\varphi}{2p_0} \\
\mathcal{D}[\alpha] &= e^{\alpha a^\dagger - \alpha^* a} & \mathcal{D}^\dagger[\alpha] a \mathcal{D}[\alpha] &= a + \alpha \\
\mathcal{D}[\alpha + \beta] &= \mathcal{D}[\beta] \mathcal{D}[\alpha] e^{(\alpha\beta^* - \alpha^*\beta)/2} & \mathcal{D}^\dagger[\alpha] a^\dagger \mathcal{D}[\alpha] &= a^\dagger + \alpha^*
\end{aligned}$$

Thermal states and the Glauber-Sudarshan P function:

$$\begin{aligned}
\rho_{\text{thermal}} &= \sum_n \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |n\rangle\langle n| & \frac{1}{\bar{n}} &= e^{\frac{\hbar\omega}{k_B T}} - 1 \\
\rho &= \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha| & P_{\text{thermal}}(\alpha) &= \frac{1}{\pi\bar{n}} e^{-\frac{|\alpha|^2}{\bar{n}}} \\
S_{\text{vN}} &= (\bar{n} + 1) \ln(\bar{n} + 1) - \bar{n} \ln(\bar{n})
\end{aligned}$$

Wavefunctions:

$$\begin{aligned}
\psi_n(x) &= \left(\frac{1}{2\pi x_0^2}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{x^2}{4x_0^2}\right) H_n\left(\frac{x}{\sqrt{2}x_0}\right) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right) \\
\tilde{\psi}_n(p) &= \left(\frac{1}{2\pi p_0^2}\right)^{\frac{1}{4}} \frac{(-i)^n}{\sqrt{2^n n!}} \exp\left(-\frac{p^2}{4p_0^2}\right) H_n\left(\frac{p}{\sqrt{2}p_0}\right) = \left(\frac{1}{\pi m\hbar\omega}\right)^{\frac{1}{4}} \frac{(-i)^n}{\sqrt{2^n n!}} \exp\left(-\frac{p^2}{2m\hbar\omega}\right) H_n\left(\frac{p}{\sqrt{m\hbar\omega}}\right)
\end{aligned}$$