Physics 17 Class Notes
Rutherford Scattering

WCC

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Setup

We will consider the scattering of an α particle (we'll leave its charge as $q_\alpha = +Z_\alpha e$ so that these results can be adapted easily to other probe particles) from an infinitely-massive point-nucleus of charge $q = +Ze$, as shown below.

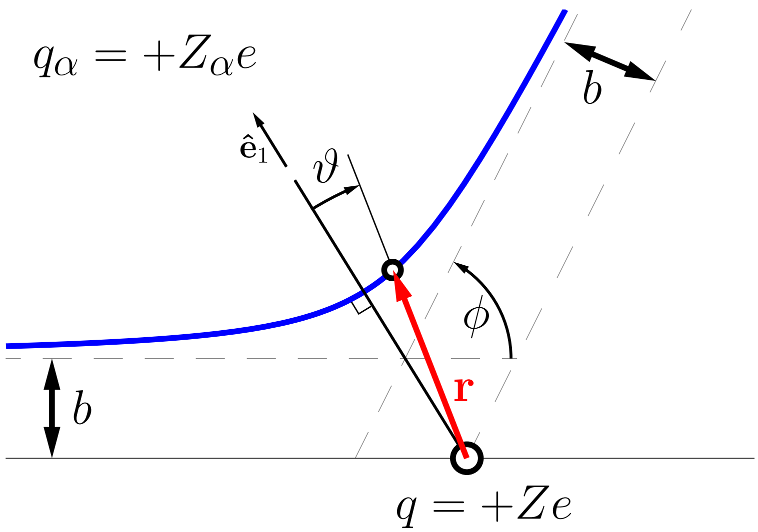

Figure 1: Geometry and definitions for Rutherford scattering.

The incoming α particle will be assumed to be non-relativistic with initial kinetic energy $K_\alpha = \frac{1}{2}m_\alpha v_\alpha^2$. No matter the initial angle of $\mathbf{v}_\alpha$ compared to the plane of the metal foil containing the nuclei, the spherical symmetry of the Coulomb potential allows us to reduce this problem to a 2D scenario as shown above. Here, b is the *impact parameter* and ϕ is the *scattering angle* (in the lab frame). Also shown are the time-dependent separation from the nucleus, $\mathbf{r}(t)$, and the angle with respect to the line of reflection symmetry, ϑ. The line of reflection symmetry is along axis $\hat{\mathbf{e}}_1$.

Sommerfeld parameter

If we consider for a moment a head-on collision (*viz.* $b = 0$), we can use conservation of energy to determine the minimum radius from the nucleus that is reached by the α, which your book calls d_{min}:

$$\begin{aligned} K_\alpha &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\frac{1}{d_{\mathrm{min}}} \\ d_{\mathrm{min}} &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\frac{1}{K_\alpha}. \end{aligned} \tag{1}$$

This radius (which is often denoted by η instead of d_{min}) is known as the (non-relativistic version of the) *Sommerfeld parameter*, and provides the scale of penetration for scattering. This can be simplified by using frequently-encountered dimensionless combination of constants that often gets its own symbol,

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0}\frac{1}{\hbar c}, \tag{2}$$

which is known as the *fine-structure constant* (you may already know that $\alpha \approx 1/137$). We can write the non-relativistic Sommerfeld parameter as

$$d_{\mathrm{min}} = Z_\alpha Z\,\alpha\frac{\hbar c}{K_\alpha}.$$

Relationship between scattering angle and impact parameter

Our first task is to find how b is related to ϕ (given K_α, Z_α, and Z). We can start by considering how much the scattering process changes the translational momentum of the α. If the initial momentum is $\mathbf{p}_\alpha$, since the scattering is elastic and the nucleus is assumed to be sufficiently heavy as to not recoil from the collision, the final momentum ($\mathbf{p}'_\alpha$) must have the same magnitude as the initial momentum, $|\mathbf{p}'_\alpha| = p'_\alpha = p_\alpha$.

If we denote the total momentum change as $\Delta\mathbf{p}_\alpha \equiv \mathbf{p}'_\alpha - \mathbf{p}_\alpha$, the graphical version of that vector equation is an isosceles triangle with legs of length p_α, apex angle ϕ, and base Δp. From this, we see that

$$\Delta p = 2p_\alpha \sin\left(\tfrac{\phi}{2}\right). \tag{3}$$

We also know that the momentum change is equal to the time integral of the net force,

$$\Delta\mathbf{p} = \int \mathrm{d}t\,\mathbf{F}(t). \tag{4}$$

For the force, we have the purely-radial (from the perspective of the nucleus) Coulomb repulsion between the stationary nucleus and the α,

$$\mathbf{F}(t) = \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\frac{1}{\left(r(t)\right)^2}\hat{\mathbf{r}}(t). \tag{5}$$

Looking at figure 1, for every position $\mathbf{r}$ for which $\vartheta = -\vartheta_0 < 0$, there is a symmetric position with $\vartheta = +\vartheta_0 > 0$ for which the components of the Coulomb force that are perpendicular to $\hat{\mathbf{e}}_1$ are equal and opposite. Since symmetry dictates that the speed of the α will be identical at these two locations, we find that the components of the force that are perpendicular to $\hat{\mathbf{e}}_1$ don't contribute to the integral (4). The direction of $\Delta\mathbf{p}$ is clearly $\hat{\mathbf{e}}_1$, and we can just work with its magnitude to write

$$\Delta p = \int \mathrm{d}t\,F(t)\cos(\vartheta(t)).$$

Next, since the force on the α is always purely radial from the nucleus, there is no torque about the nucleus, and we therefore demand that angular momentum about the nucleus be conserved at all times during the collision. Taking the direction of the α's angular momentum about the nucleus to be into the page, we can write its initial angular momentum as

$$L_\alpha = b\, p_\alpha. \tag{6}$$

At other locations along the trajectory, we can relate it to $\vartheta(t)$ via

$$\begin{aligned} L(t) &= \ |\mathbf{r}(t) \times \mathbf{p}(t)| \\ &= m\,|\mathbf{r}(t) \times \mathbf{v}(t)| \end{aligned}$$

Using our (r, ϑ) system in figure 1 as a plane-polar coordinate system, we would write the velocity in these coordinates in the form

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\vartheta}\,\hat{\boldsymbol{\vartheta}}. \tag{7}$$

Since $\mathbf{r}$ is always perpendicular to $\hat{\boldsymbol{\vartheta}}$, we see that

$$|\mathbf{r}(t) \times \mathbf{v}(t)| = r\,r\dot{\vartheta}$$

and therefore

$$\begin{aligned} L_\alpha &= m\,(r(t))^2\,\frac{\mathrm{d}\vartheta}{\mathrm{d}t} \\ \frac{\mathrm{d}t}{(r(t))^2} &= \frac{m}{L_\alpha}\,\mathrm{d}\vartheta. \end{aligned}$$

Looking back at Eq. (5), we see that we can use this to reqrite the integral over t as an integral over ϑ:

$$\begin{aligned} \Delta p &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0} \int \frac{\mathrm{d}t}{(r(t))^2}\,\cos(\vartheta(t)) \\ &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\,\frac{m}{L_\alpha} \int \mathrm{d}\vartheta\,\cos(\vartheta) \end{aligned}$$

For the limits of this integral, we note that the entire collision spans a range of $\Delta\vartheta = \pi - \phi$ and is symmetric about $\vartheta = 0$, so the integral goes from $\vartheta_{\mathrm{min}} = \frac{\phi}{2} - \frac{\pi}{2}$ to $\vartheta_{\mathrm{max}} = \frac{\pi}{2} - \frac{\phi}{2}$,

$$\begin{aligned} \Delta p &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\,\frac{m}{L_\alpha}\left[\sin(\tfrac{\pi}{2} - \tfrac{\phi}{2}) - \sin(\tfrac{\phi}{2} - \tfrac{\pi}{2})\right] \\ &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\,\frac{m}{L_\alpha}\left[\cos(\tfrac{\phi}{2}) + \cos(\tfrac{\phi}{2})\right] \\ &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\,\frac{2m}{b\,p_\alpha}\cos(\tfrac{\phi}{2}). \end{aligned}$$

Going back to Eq. (3) allows us to write this expression as

$$2p_\alpha \sin(\tfrac{\phi}{2}) = \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\,\frac{2m}{b\,p_\alpha}\cos(\tfrac{\phi}{2})$$

and solve for $b(\phi)$,

$$\begin{aligned} b &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\,\frac{2m}{2\,p_\alpha^2}\,\frac{\cos(\frac{\phi}{2})}{\sin(\frac{\phi}{2})} \\ &= \frac{Z_\alpha Z e^2}{4\pi\epsilon_0}\,\frac{1}{2K_\alpha}\,\frac{1}{\tan(\frac{\phi}{2})}. \end{aligned}$$

The non-relativistic Sommerfeld parameter allows us to write the relationship between the impact parameter and the scattering angle as

$$b = \frac{d_{\mathrm{min}}}{2}\,\frac{1}{\tan(\frac{\phi}{2})}.$$