Spherical Vector Cheat Sheet

WCC

$$\hat{\mathbf{e}}_1 = -\tfrac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \qquad \hat{\mathbf{e}}_1^* = -\tfrac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \qquad \hat{\mathbf{e}}_1^* = -\,\hat{\mathbf{e}}_{-1} \qquad \hat{\mathbf{x}} = -\tfrac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_{-1})$$

$$\hat{\mathbf{e}}_0 = \hat{\mathbf{z}} \qquad \hat{\mathbf{e}}_0^* = \hat{\mathbf{z}} \qquad \hat{\mathbf{e}}_0^* = \hat{\mathbf{e}}_0 \qquad \hat{\mathbf{y}} = \tfrac{i}{\sqrt{2}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_{-1})$$

$$\hat{\mathbf{e}}_{-1} = \tfrac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \qquad \hat{\mathbf{e}}_{-1}^* = \tfrac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \qquad \hat{\mathbf{e}}_{-1}^* = -\,\hat{\mathbf{e}}_1 \qquad \hat{\mathbf{z}} = \hat{\mathbf{e}}_0$$

$$\hat{\mathbf{e}}_q \cdot \hat{\mathbf{e}}_q = \delta_{q0} \qquad \hat{\mathbf{e}}_p^* \cdot \hat{\mathbf{e}}_q = \delta_{pq} \qquad \hat{\mathbf{e}}_q \times \hat{\mathbf{e}}_q = 0 \qquad \hat{\mathbf{e}}_{\pm 1}^* \times \hat{\mathbf{e}}_{\pm 1} = \pm i\hat{\mathbf{e}}_0$$

This one always creeps me out at first $\rightarrow$ $\hat{\mathbf{e}}_0 \times \hat{\mathbf{e}}_{\pm 1} = \mp i\hat{\mathbf{e}}_{\pm 1}$ $\qquad$ $\hat{\mathbf{e}}_{\mp 1} \times \hat{\mathbf{e}}_{\pm 1} = \mp i\hat{\mathbf{e}}_0$

$$\mathbf{A} = \sum_q A_q\,\hat{\mathbf{e}}_q^* = \sum_q (-)^q A_q\,\hat{\mathbf{e}}_{-q} = \sum_q \left(\mathbf{A} \cdot \hat{\mathbf{e}}_q\right)\hat{\mathbf{e}}_q^* = (A_1,\, A_0,\, A_{-1}) \qquad A_q = \mathbf{A} \cdot \hat{\mathbf{e}}_q$$

Note the following subtlety of the unit vector notation in light of the above definition: $\hat{\mathbf{e}}_q$ is a *vector*, not a *component of* a vector. For each of the three values of q, $\hat{\mathbf{e}}_q$ has three components; in the spherical basis, they're somewhat non-obvious. For example, $\hat{\mathbf{e}}_1 = (0, 0, -1)$ has only one nonzero component in the spherical basis . . . and it's the $q = -1$ component (and it's -1)!

$$\mathbf{A} \cdot \mathbf{B} = \sum_q (-)^q A_q\, B_{-q} = -\sqrt{3}\; T_0^{(0)}[\mathbf{A}, \mathbf{B}] = A_x B_x + A_y B_y + A_z B_z$$

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= \textstyle\sum_{ijk} \epsilon_{ijk}\, \hat{\mathbf{r}}_i\, A_j B_k = (A_y B_z - A_z B_y)\,\hat{\mathbf{x}} + (A_z B_x - A_x B_z)\,\hat{\mathbf{y}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{z}} \\
&= -\,i\sqrt{2}\; T^{(1)}[\mathbf{A}, \mathbf{B}] = -i(A_1 B_0 - A_0 B_1)\,\hat{\mathbf{e}}_1^* + i(A_{-1} B_1 - A_1 B_{-1})\,\hat{\mathbf{e}}_0^* - i(A_0 B_{-1} - A_{-1} B_0)\,\hat{\mathbf{e}}_{-1}^* \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -i \begin{vmatrix} \hat{\mathbf{e}}_{-1}^* & -\hat{\mathbf{e}}_0^* & \hat{\mathbf{e}}_1^* \\ A_1 & A_0 & A_{-1} \\ B_1 & B_0 & B_{-1} \end{vmatrix}
\end{aligned} \tag{1}$$

Pure σ^+ polarization: $\hat{\boldsymbol{\epsilon}}_{\sigma^+} = \;-1\,\hat{\mathbf{e}}_{-1}^* = (0, 0, -1) = \hat{\mathbf{e}}_1$ $\qquad$ $\hat{\boldsymbol{\epsilon}}_{\sigma^+}^* = \hat{\mathbf{e}}_1^* = (1, 0, 0) = -\hat{\mathbf{e}}_{-1}$

Pure σ^- polarization: $\hat{\boldsymbol{\epsilon}}_{\sigma^-} = \;-1\,\hat{\mathbf{e}}_1^* = (-1, 0, 0) = \hat{\mathbf{e}}_{-1}$ $\qquad$ $\hat{\boldsymbol{\epsilon}}_{\sigma^-}^* = \hat{\mathbf{e}}_{-1}^* = (0, 0, 1) = -\hat{\mathbf{e}}_1$

Circularity of polarization: $C \equiv -i\,(\hat{\boldsymbol{\epsilon}}^* \times \hat{\boldsymbol{\epsilon}}) \cdot \hat{\mathbf{k}}$

This ranges from $C = -1$ for RCP to $C = 0$ for any linear polarization to $C = +1$ for LCP (where I am using the Jackson convention for naming right and left circular polarizations). If the beam propagates along $\hat{\mathbf{k}} \parallel +\hat{\mathbf{z}}$, RCP is σ^- and LCP is σ^+.