

Æ codes

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(Dated: March 5, 2024)

A. Non-locality of rigid body shift noise

Consider the case of a planar rotor [1, Sec. IV], relevant to the case of a heteronuclear diatomic molecule confined to rotate in a plane. The Hilbert space is described by the angular position states $\{|\phi\rangle : \phi \in [0, 2\pi)\}$, or equivalently, the angular momentum states $\{|\ell\rangle : \ell \in \mathbb{Z}\}$.

A natural decomposition of any trace-class operator $\hat{O}$ acting on the planar rotor space is

$$\hat{O} = \sum_{\ell \in \mathbb{Z}} \int_0^{2\pi} d\phi c_\ell(\phi) e^{i\ell\hat{\theta}} e^{-i\phi\hat{L}}. \quad (\text{A1})$$

The position shift operators $e^{-i\phi\hat{L}} = \int_0^{2\pi} d\phi' |\phi + \phi'\rangle \langle \phi'|$ and the angular momentum shift operators $e^{i\ell\hat{\theta}} = \sum_{\ell' \in \mathbb{Z}} |\ell + \ell'\rangle \langle \ell'|$ form a complete basis of operators in this space.

Let us apply the above decomposition to a general angular momentum transition,

$$\hat{\sigma}_{mn} = |m\rangle \langle n| = \sum_{\ell \in \mathbb{Z}} \int_0^{2\pi} d\phi c_\ell(\phi) e^{i\ell\hat{\theta}} e^{-i\phi\hat{L}} \text{ with} \quad (\text{A2})$$

$$c_\ell(\phi) = \delta_{\ell, m-n} \frac{e^{in\phi}}{2\pi}. \quad (\text{A3})$$

The weight of the ϕ -position shift $|c_{m-n}(\phi)|$ is independent of ϕ i.e. there is uniform, non-zero contribution from *all* angular position shift operators. Thus, a transition noise operator does not only contain small shifts in the angular position, but contains equal contributions from shifts of all possible amplitudes. In particular, the noise operators contains significant contributions from large angular position shifts and is hence, *non-local* in the phase space of the rotor.

Similar (but no longer orthonormal [1]) decomposition into angular position shifts ($\hat{X}_S$) and angular momentum shifts ($\hat{Y}_m^J$) is possible for the general diatomic molecule. We describe one such transition operator here:

$$|J, m-1\rangle \langle J, m| = \int_{SO(3)} d\mu(S) \sum_{\substack{J' \in \mathbb{Z}^+ \\ -J' \leq m' \leq J'}} \mathcal{E}_{m'}^{J'}(S) \hat{Y}_{m'}^{J'} \hat{X}_S \text{ with} \quad (\text{A4})$$

$$\mathcal{E}_{m'n'}^{J'}(S) = \sqrt{\frac{2J'+1}{4\pi}} D_{m-1, m-m'}^J(S) C_{J', m'; J, m-m'}^{J, m\star} C_{J', 0; J, 0}^{J, 0\star}. \quad (\text{A5})$$

The actions of position shift operators $\hat{X}_S$ and the angular momentum shift operators $\hat{Y}_m^J$ are defined by their action on the position basis states $|v\rangle$

$$\hat{X}_S |v\rangle = |Sv\rangle \quad (\text{A6})$$

$$\hat{Y}_m^J |v\rangle = Y_m^J(v) |v\rangle, \quad (\text{A7})$$

where $Y_m^J(v)$ is the spherical harmonic evaluated at v , $C_{c,d;e,f}^{a,b}$ are the Clebsch-Gordan coefficients, and $\mu(S)$ is the Haar measure on $SO(3)$. The actual order- r transition noise operators as described in the main text are

$$\hat{K}_{\delta m=-1}^{r(\delta J=0)} \propto \sum_{|m| \leq J} C_{J, m; r, -1}^{J, m-1} |J, m-1\rangle \langle J, m|. \quad (\text{A8})$$

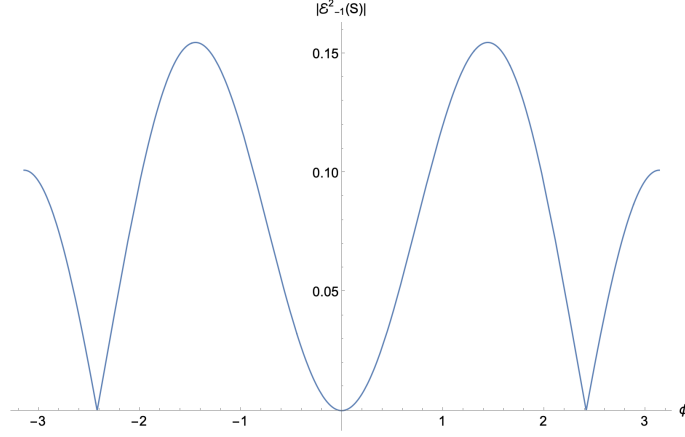


FIG. A.1. The weights $|\mathcal{E}_{-1}^2(S)|$ for the order-2 transition operator $\hat{K}_{-1}^{2(0)}$ acting on $J = 2$ level of the system as a function of the angle of rotation ϕ about the axis $(1, 0, 0)$ which is perpendicular to the symmetry axis of the molecule. The angle $\phi = 0$ represents zero shift, and $\phi = \pm\pi$ is the maximum possible shift in the angular orientation of the rotor.

The relevant weights of the shift operators take the form

$$\mathcal{E}_{m'}^{J'}(S) \propto \sum_{|m| \leq J_0} \sqrt{\frac{2J' + 1}{4\pi}} C_{J,m;-1}^{J,m-1} C_{J',m';J,m-m'}^{J,m*} C_{J',0;J,0}^{J,0*} D_{m-1,m-m'}^J(S). \quad (\text{A9})$$

As an example, Fig. A.1 shows the absolute value of the weights $|\mathcal{E}_{-1}^2(S)|$ for the order-2 transition operator acting on $J = 2$ level of the system as a function of the angular shift about an axis perpendicular to the axis of the molecule. These weights reveal that the transition noise operators for the general diatomic molecule also contain significant contributions from large shifts in the angular orientation, causing molecular codes [1] to fail against them.

B. No-go theorem

We prove the following no-go theorem: *Given a Hilbert space with countable basis and orthogonal vectors $|\psi\rangle$ and $|\phi\rangle$, an error channel containing the Kraus operator $\hat{E} = |\phi\rangle\langle\psi|$ is correctable only if the codespace is not supported by $|\psi\rangle$.*

Proof. Proof is simple by checking the KL conditions for this error operator. Consider the two general encoded states

$$|\bar{0}\rangle = \sum_i \alpha_i |\psi_i\rangle \quad \text{and} \quad |\bar{1}\rangle = \sum_i \beta_i |\psi_i\rangle, \quad (\text{B10})$$

with the sum running over the Hilbert space basis $\{|\psi_i\rangle\}$ and set $|\psi_0\rangle = |\psi\rangle$ without loss of generality. To be valid code states, the following conditions need to be satisfied simultaneously given $\hat{E}^\dagger \hat{E} = |\psi_0\rangle\langle\psi_0|$.

$$|\alpha_0|^2 = \langle \bar{0} | \hat{E}^\dagger \hat{E} | \bar{0} \rangle \stackrel{!}{=} \langle \bar{1} | \hat{E}^\dagger \hat{E} | \bar{1} \rangle = |\beta_0|^2 \quad (\text{B11})$$

$$\langle \bar{0} | \hat{E}^\dagger \hat{E} | \bar{1} \rangle = \beta_0 \alpha_0^* \stackrel{!}{=} 0, \quad (\text{B12})$$

where we denote satisfaction by the symbol “ $\stackrel{!}{=}$ ”. The only solution to Eqn. B11 and Eqn. B12 is $\alpha_0 = \beta_0 = 0$ i.e. the code states having zero support over the state $|\psi\rangle$. $\square$

C. Protection against dephasing

We concentrate on identifying code states designed to protect against dephasing errors up to order k . These code states are confined to a single J manifold, and we assume a separation such that the off-diagonal KL conditions (i.e. $i \neq j$) are trivially fulfilled. The remaining KL constraints are

$$\langle \bar{0} | \hat{m}^r | \bar{0} \rangle - \langle \bar{1} | \hat{m}^r | \bar{1} \rangle = 0 \quad (\text{C13})$$

for all r upto $2n$. Considering the general code states

$$|\bar{0}\rangle = \sum_m \alpha_m |J, m\rangle \quad \text{and} \quad |\bar{1}\rangle = \sum_m \beta_m |J, m\rangle, \quad (\text{C14})$$

these constraints simply become linear equations

$$\left\{ \sum_{m=-J}^J (|\alpha_m|^2 - |\beta_m|^2) m^r = 0 \right\}_{0 \leq r \leq 2n}. \quad (\text{C15})$$

This is a system of $2n$ linear equations in the $4J + 2$ variables $|\alpha_i|^2, |\beta_j|^2$. One might initially consider the solution $\alpha_i = \beta_i \forall i$, but quickly realizes that it fails to meet the necessary requirements for our ansatz and is not valid. Due to the constraints on the ansatz, the actual number of variables involved is generally much smaller than $4J + 2$. To illustrate, we explicitly outline the system of equations for spontaneous decay arising from the symmetric ansatz.

$$|\bar{0}\rangle = \alpha_{-m_1} |J, -m_1\rangle + \alpha_{m_1} |J, m_1\rangle \quad \text{and} \quad |\bar{1}\rangle = \beta_{-m_2} |J, -m_2\rangle + \beta_0 |J, 0\rangle + \beta_{m_2} |J, m_2\rangle. \quad (\text{C16})$$

Here, instead of $4J + 2$ variables, there are only 5, independent of J (provided $J \geq 6$). The constraints to be satisfied are

$$\{ |\alpha_{-m_1}|^2 (-m_1)^i + |\alpha_{m_1}|^2 m_1^i - |\beta_0|^2 \delta_{0,i} - |\beta_{-m_2}|^2 (-m_2)^i - |\beta_{m_2}|^2 m_2^i = 0 \}_{0 \leq i \leq 2}. \quad (\text{C17})$$

This is a system of *three* linear equations in *five* variables admitting infinitely many solutions, one of which is presented as the symmetric $n = 1$ code in the main text.

Our method enables the construction of such error-correcting code families for protection against errors of any order. We illustrate with a countersymmetric code family, designed to correct upto order 2 errors.

$$\begin{aligned} |\bar{0}\rangle &= \sqrt{\frac{m_1(m_1-m_2)m_2}{(m_1-m_3)(m_1-m_2+m_3)(m_2+m_3)}} |J, -m_3\rangle + \sqrt{\frac{m_2(m_2-m_3)m_3}{(m_1+m_2)(m_1-m_3)(m_1-m_2+m_3)}} |J, -m_1\rangle \\ &\quad + \sqrt{\frac{m_1 m_3 (m_1+m_3)}{(m_1+m_2)(m_1-m_2+m_3)(m_2+m_3)}} |J, m_2\rangle \\ |\bar{1}\rangle &= \sqrt{\frac{m_1 m_3 (m_1+m_3)}{(m_1+m_2)(m_1-m_2+m_3)(m_2+m_3)}} |J, -m_2\rangle + \sqrt{\frac{m_2(m_2-m_3)m_3}{(m_1+m_2)(m_1-m_3)(m_1-m_2+m_3)}} |J, m_1\rangle \\ &\quad + \sqrt{\frac{m_1(m_1-m_2)m_2}{(m_1-m_3)(m_1-m_2+m_3)(m_2+m_3)}} |J, m_3\rangle \end{aligned} \quad (\text{C18})$$

where $m_3 - m_1, m_2 - m_1 \geq 5$ and $m_1 \geq 2$. Lastly, our approach is not limited to systems with inherent symmetries; it can be applied to construct codes using any ansatz that meets the separation requirements. As a comprehensive illustration, we showcase the general compact code capable of correcting up to order 1 errors.

$$\begin{aligned} |\bar{0}\rangle &= \sqrt{\frac{(m_3-m_2)(m_4-m_2)}{(m_2-m_1)(m_1+m_2-m_3-m_4)}} |J, m_1\rangle + \sqrt{\frac{(m_1-m_3)(m_1-m_4)}{(m_1-m_2)(m_1+m_2-m_3-m_4)}} |J, m_2\rangle \\ |\bar{1}\rangle &= \sqrt{\frac{(m_1-m_4)(m_2-m_4)}{(m_1+m_2-m_3-m_4)(m_3-m_4)}} |J, m_3\rangle + \sqrt{\frac{(m_1-m_3)(m_3-m_2)}{(m_1+m_2-m_3-m_4)(m_3-m_4)}} |J, m_4\rangle. \end{aligned} \quad (\text{C19})$$

where the separation requirement are met if $|m_i - m_j| \geq 3 \forall i \neq j$. This family admits the counter-symmetric $n = 1$ code as a member upon setting $m_1 = -m_1, m_3 = -m_2$ and $m_4 = m_1$.

D. Theorems concerning $\hat{Y}$ matrix elements

Using an ansatz with sufficient separation between supported states, the only KL conditions for higher order transition errors which are not trivially satisfied include terms of the form $\langle J_0, m | \hat{K}^{r_1 \dagger} \hat{K}^{r_2} | J_0, m \rangle$ (see main text for definition of $\hat{K}^r$) which are proportional to products of $\hat{Y}$ operators $\langle J_0 + \delta J, m + \delta m | \hat{Y}_{\delta m}^{r_1} | J_0, m \rangle \langle J_0 + \delta J, m + \delta m | \hat{Y}_{\delta m}^{r_2} | J_0, m \rangle$. Then, the following claim is adequate to prove that such KL conditions reduce to the KL conditions for the generalized dephasing operator set $\{\hat{m}^k | 0 \leq k \leq n\}$ for any $n \geq \max(r_1, r_2)$:

Claim D.1. $\langle J_0 + \delta J, m + \delta m | \hat{Y}_{\delta m}^{r_1} | J_0, m \rangle \langle J_0 + \delta J, m + \delta m | \hat{Y}_{\delta m}^{r_2} | J_0, m \rangle = \text{poly}(r_1 + r_2, m)$ where $\text{poly}(r, m)$ denotes a general polynomial of degree r in variable m and $\hat{Y}$ matrix elements are as defined in [1, TABLE V].

We note some other results required to prove the stated claim. The matrix elements of $\hat{Y}$ can be written as a product of Clebsch-Gordan (CG) coefficients

$$\langle J, m + \delta m | \hat{Y}_{\delta m}^r | J_0, m \rangle = \sqrt{\frac{(2r+1)(2J_0+1)}{4\pi(2J+1)}} C_{r, \delta m; J_0, m}^{J, m + \delta m} C_{r, 0; J_0, 0}^{J, 0} \propto C_{J_0, m; r, \delta m}^{J, m + \delta m} \quad (\text{D20})$$

where the m dependence lies entirely in one CG coefficient.

Observation D.2. For integer or half integer J_0 , all Clebsch-Gordan coefficients of the form $C_{J_0, m; r, \pm r}^{J, m \pm r}$ and $C_{J_0, m; r, \delta m}^{J, \pm r, m + \delta m}$ are square roots of real polynomials of degree $2r$ in m .

This observation is verified simply by looking at the expressions for the mentioned CG coefficients [2, Chapter 8].

Theorem D.3. For $r \geq |\delta m|$, $C_{r, \delta m; J_0, m}^{J, m + \delta m} = \text{poly}(m, r - |\delta m|) C_{|\delta m|, \delta m; J_0, m}^{J, m + \delta m}$.

Proof. Note the identity involving Clebsch-Gordan coefficients (8.6.5 [2])

$$\begin{aligned} C_{J_0, m; r, \delta m}^{J, m + \delta m} &= \left[2m + \frac{\delta m}{r(r+1)} \{r(r+1) + J_0(J_0+1) - J(J+1)\} \right]^{-1} \\ &\quad \left[\frac{1}{r(2r+1)} \{ (r^2 - \delta m^2) (-r + J_0 + J + 1)(r - J_0 + J)(r + J_0 - J)(r + J_0 + J + 1) \}^{\frac{1}{2}} C_{J_0, m; r-1, \delta m}^{J, m + \delta m} + \right. \\ &\quad \left. \frac{1}{(r+1)(2r+1)} \{ [(r+1)^2 - \delta m^2] (-r + J_0 + J)(r - J_0 + J + 1)(r + J_0 - J + 1)(r + J_0 + J + 2) \}^{\frac{1}{2}} \right. \\ &\quad \left. C_{J_0, m; r+1, \delta m}^{J, m + \delta m} \right]. \end{aligned} \quad (\text{D21})$$

For the sake of clarity, we substitute $C_{J_0, m; r, \delta m}^{J, m + \delta m}$ with $C_r(m)$ to label the rank r symbol as a function of m . Then,

$$C_r(m) = \text{poly}_1(m, 1) C_{r-1}(m) + \text{poly}_2(m, 0) C_{r-2}(m). \quad (\text{D22})$$

We also observe that for every $\text{poly}_1, \text{poly}_2$, the product of the two is some poly_3 such that $\text{poly}_1(m, a) \text{poly}_2(m, b) = \text{poly}_3(m, a + b)$. Using these relations, we inductively prove our theorem. The statement to prove is that for every $r \geq |\delta m|$, $C_r(m) = \text{poly}(m, r - |\delta m|) C_{|\delta m|}(m)$.

Base case: $C_{|\delta m|} = C_{|\delta m|}(m) = \text{poly}(m, 0) C_{|\delta m|}$ and $C_{|\delta m|+1}(m) = \text{poly}(m, 1) C_{|\delta m|}(m)$ using Eq. (D22).

Inductive step: Assume that $C_{r-1}(m) = \text{poly}_1(m, r - 1 - |\delta m|) C_{|\delta m|}(m)$ and $C_{r-2}(m) = \text{poly}_2(m, r - 2 - |\delta m|) C_{|\delta m|}(m)$, then

$$\begin{aligned} C_r(m) &= \tilde{\text{poly}}_1(m, 1) C_{r-1}(m) + \tilde{\text{poly}}_2(m, 0) C_{r-2}(m) \\ &= \left(\tilde{\text{poly}}_1(m, 1) \text{poly}_1(m, r - 1 - |\delta m|) + \tilde{\text{poly}}_2(m, 0) \text{poly}_2(m, r - 2 - |\delta m|) \right) C_{|\delta m|}(m) \\ &= \text{poly}(m, r - |\delta m|) C_{|\delta m|}(m). \end{aligned} \quad (\text{D23})$$

Combining the base case and the inductive step, the statement is proved to hold for $r \geq |\delta m|$. $\square$

Theorem D.4. All Clebsch-Gordan coefficients $C_{r, \delta m; J_0, m}^{J, m + \delta m}$ can be formulated as square roots of real polynomials of degree $2r$ in m .

Proof. For clarity, we will denote by $C_r^{\delta J, \delta m}(m)$ the Clebsch-Gordan coefficient $C_{J_0, m; r, \delta m}^{J_0 + \delta J, m + \delta m}$ as a function of m . The theorem can be proved by induction. The statement to prove for all CG coefficients is $C_r^{\delta J, \delta m}(m) = \sqrt{\text{poly}(m, 2r)}$.

Base case: $C_0^{\delta J=0, \delta m=0}(m) = \sqrt{\text{poly}(m, 0)}$ by observation D.2.

Inductive step: Assume that $C_k^{\delta J, \delta m}(m) = \sqrt{\text{poly}(m, 2k)} \forall |\delta J|, |\delta m| \leq k \forall k$ upto $r - 1$. First, consider the coefficients $C_r^{\delta J=\pm r, \delta m}(m)$ and $C_r^{\delta J, \delta m=\pm r}(m)$. These satisfy the statement by use of Observation D.2. Next, consider all other coefficients $C_r^{\delta J, \delta m}(m)$ such that $\delta J \neq \pm r, \delta m \neq \pm r$. Then using Theorem D.3 and Observation D.2, these satisfy

$$\begin{aligned} C_r^{\delta J, \delta m}(m) &= \sqrt{\text{poly}(m, 2r - 2|\delta m|)} C_{|\delta m|}^{\delta J, \delta m} \\ &= \sqrt{\text{poly}(m, 2r)}. \end{aligned} \quad (\text{D24})$$

Combining the base case and the inductive step, the statement is proved to hold for all CG coefficients. $\square$

We are finally ready to prove claim D.1.

Proof. With the use the stated theorems, the product of the $\hat{Y}$ matrix elements in claim D.1 simplifies as

$$\begin{aligned}
 \langle J_0 + \delta J, m + \delta m | \hat{Y}_{\delta m}^{r_1} | J_0, m \rangle \langle J_0 + \delta J, m + \delta m | \hat{Y}_{\delta m}^{r_2} | J_0, m \rangle &= poly_1(m, r_1 - |\delta m|) \\
 &\times poly_2(m, r_2 - |\delta m|) \\
 &\times \left(\sqrt{poly_3(m, 2\delta m)} \right)^2 \\
 &= poly(m, r_1 + r_2).
 \end{aligned} \tag{D25}$$

□

E. Binomial like $\mathbb{A}\mathbb{E}$ codes

Here, we show that the binomial-like qudit codestates, as formulated in the main text protect against dephasing errors upto order n . It is sufficient to show $\langle \bar{0} | \hat{m}^r | \bar{0} \rangle - \langle \bar{1} | \hat{m}^r | \bar{1} \rangle = 0 \forall 0 \leq r \leq 2n$:

$$\begin{aligned}
 \langle \bar{0} | \hat{m}^k | \bar{0} \rangle - \langle \bar{1} | \hat{m}^k | \bar{1} \rangle &= \sum_{\text{even } k=0}^{2n+1} \binom{2n+1}{k} [-m_0 + k(2n+1)]^r - \sum_{\text{odd } k=0}^{2n+1} \binom{2n+1}{k} [-m_0 + k(2n+1)]^r \\
 &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k [-m_0 + k(2n+1)]^r \\
 &= \sum_{l=0}^r \binom{r}{l} (-m_0)^{r-l} (2n+1)^l \left\{ \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k k^l \right\}
 \end{aligned} \tag{E26}$$

where it is a well known binomial identity that the sum in the curly braces is zero $\forall 0 \leq l \leq 2n$ i.e. $\forall 0 \leq r \leq 2n$, the entire sum vanishes.

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